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The Shuman Filtering Operator and the Numerical Computation of Shock Waves

A. C. VLIEGENTHART

Department of Mathematics, Delft University of Technology, The Netherlands

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SUMMARY
Shuman's method of filtering short-wave components is discussed. In certain cases, this method can be used to prevent nonlinear instabilities in numerical calculations. As illustrative example, the computation of a detached shock in front of a blunt body is considered.

1. Introduction

For numerical work in weather prediction Shuman [1] devised a method of filtering short-wave components.

The application of dissipative finite-difference schemes for the numerical computation of flowfields containing shock waves, yields a damping of short Fourier components. However, the damping is absent for selective components of the solution. These encountered oscillations, which have no physical meaning, can be suppressed by applying Shuman's method. In certain cases this even appears to remove nonlinear instabilities as with the computation of a detached shock in front of a blunt body. This problem has been solved numerically by Bohachevsky and Rubin [2], Burstein [3] and Lapidus [4].

Bohachevsky and Rubin used Lax's scheme which has first-order accuracy and generally does not show nonlinear instabilities. However, it yields a shock which is spread out over a large number of mesh spacings. The scheme obtained by Lax and Wendroff has been applied by Burstein. To prevent nonlinear instabilities, an artificial viscosity term is added here.

Lapidus described the smoothing of short components by means of a finite-difference smoothing operator which is not an integral part of the difference approximation.

The complicacy of the second-order accurate methods used by Burstein and Lapidus makes them laborious to program. Our objective was to devise a method which has the simplicity of Lax's method, but yields a shock which is considerably narrower.

2. One-Dimensional Considerations

Consider the system of equations of the form

$$\boldsymbol{u}_t+\boldsymbol{f}_x=\boldsymbol{0}\,, \qquad -\infty<x<+\infty\,,\ t\geqq 0\,, \tag{2.1}$$

where $\boldsymbol{u}$ is a p-component vector function of x and t, and $\boldsymbol{f}(\boldsymbol{u})$ is a known vector function of $\boldsymbol{u}$.

After differentiation we obtain the quasi-linear system $\boldsymbol{u}_t+A(\boldsymbol{u})\boldsymbol{u}_x=\boldsymbol{0}$, with $A=\operatorname{grad}\boldsymbol{f}$. The system will be hyperbolic, i.e., the matrix A has p real and different eigenvalues for all values of $\boldsymbol{u}$.

The numerical integration of system (2.1) will be carried out by means of explicit difference schemes. To this end an orthogonal mesh system with constant mesh spacings Δx and Δt is imposed on the half plane $-\infty<x<+\infty$, $t\geqq 0$.

A difference approximation to system (2.1) which has been introduced by Richtmyer [5] is

$$\begin{aligned} \boldsymbol{u}_j^{l+1} &= \tfrac{1}{2}(\boldsymbol{u}_{j+1}^l+\boldsymbol{u}_{j-1}^l)-\tfrac{1}{2}\lambda(\boldsymbol{f}_{j+1}^l-\boldsymbol{f}_{j-1}^l) \\ \boldsymbol{u}_j^{l+2} &= \boldsymbol{u}_j^l-\lambda(\boldsymbol{f}_{j+1}^{l+1}-\boldsymbol{f}_{j-1}^{l+1})\,, \end{aligned} \tag{2.2}$$

where $\boldsymbol{u}(j\Delta x, l\Delta t)=\boldsymbol{u}_j^l$, and $\lambda=\Delta t/\Delta x$.

The provisional values at the time $t=(l+1)\Delta t$ are obtained from the first step, whilst the second step yields the final values at $t=(l+2)\Delta t$. It is well known that this scheme is consistent with system (2.1) and has a truncation error which is $0[(\Delta t)^3]$ in the smooth part of the solution. We suppose that $\lambda=0(1)$.

If $\boldsymbol{u}(x, t)$ is periodic in x, then substitution of the component $\hat{\boldsymbol{u}}_{(q)}e^{iqx}$ of the Fourier series for $\boldsymbol{u}(x, t)$ yields in the case of a constant matrix A

$$\hat{\boldsymbol{u}}_{(q)}^{l+2} = G(\Delta t, q)\hat{\boldsymbol{u}}_{(q)}^l ,$$

with

$$G(\Delta t, q) = I - i\lambda A \sin(2\xi) - \lambda^2 A^2 \{1-\cos(2\xi)\} . \tag{2.3}$$

For the derivation of relation (2.3) see [6].

G is called the amplification matrix, I is the unit matrix, whilst $\xi=q\Delta x$.

If the eigenvalues of matrix A are denoted by a_ν, then the eigenvalues g_ν of matrix G are

$$g_\nu(\Delta t, q) = 1 - i\mu \sin(2\xi) - \mu^2\{1-\cos(2\xi)\} ,$$

where $\mu=\lambda a_\nu$, $\nu=1, 2, \ldots, p$.

In [7] we obtained the necessary and sufficient stability condition for scheme (2.2):

$$|\mu| \leqq 1 ,$$

and derived

$$|g_\nu|^2 = 1-4\mu^2(1-\mu^2)\sin^4\xi . \tag{2.4}$$

From the last relation we concluded that Richtmyer's scheme (2.2) is dissipative of fourth order, provided that $|\mu|<1$. For definitions of stability and dissipation see [8].

Scheme (2.2) has no dissipation in three cases (see (2.4)):

(1) $|\mu|=1$;
(2) $\mu=0$;
(3) $\sin\xi=0$.

If the wavelength is denoted by L, then we may conclude from the relation $L/\Delta x=2\pi/\xi$, that case (3) is present if, for instance, $L=2\Delta x$, representing the length of the longest wave which is not damped by our difference approximation.

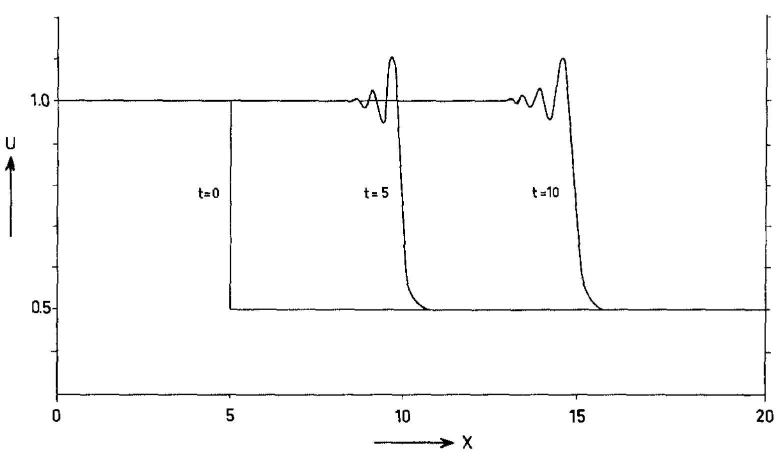

Figure 1. Shock calculation with equation $u_t+u_x=0$ and Richtmyer's scheme; $\Delta x=0.04$, $\Delta t=0.02$.

Figure 1 shows the initial values and solutions at the times $t=5$ and $t=10$ of the equation $u_t+u_x=0$, which is a special form of system (2.1). The amplified short-wave pulses can clearly be observed.

These oscillations can be damped by the following smoothing element described by Shuman [1]:

$$u_j'^{l+2} = \tfrac{1}{4}(u_{j+1}^{l+2} + 2u_j^{l+2} + u_{j-1}^{l+2}) . \tag{2.5}$$

Substitution of the Fourier component $\hat{u}_{(q)} e^{iqx}$ into (2.5) yields

$$\hat{u}_{(q)}'^{l+2} = \tfrac{1}{2}(1+\cos\xi)\,\hat{u}_{(q)}^{l+2} . \tag{2.6}$$

The multiplicative factor of (2.6) is real, so that the phases of the components are unaffected. However damping occurs for the selected components with about $\cos\xi = -1$, or, for instance, again $L = 2\Delta x$.

Because of the linearity of (2.5) the factor $\frac{1}{2}(1+\cos\xi)$ is independent of j.

It is clear that the mean value of a field of infinite extent is not affected by (2.5) so that the conservation (see [8]) is still present.

By means of a Taylor expansion about $x = j\Delta x$ we obtain from (2.5):

$$u' = u + \tfrac{1}{4}(\Delta x)^2 u_{xx} + 0[(\Delta x)^4] .$$

Thus the element (2.5) yields dissipation to our difference approximation.

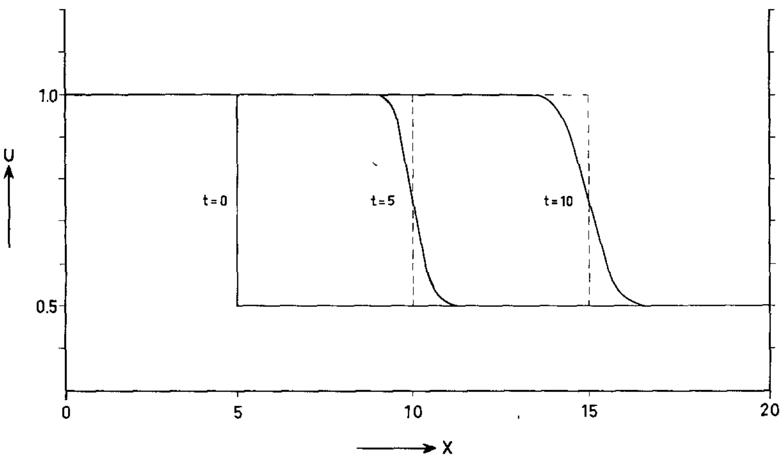

Figure 2. Similar to Fig. 1, except that smoothing has been applied. ------ exact solution.

Figure 2 is similar to Figure 1, except that the operator (2.5) was applied after every two steps of scheme (2.2).

In the nonlinear case not all oscillations may be damped out completely, as the example in the next section shows.

The three-step scheme consisting of the combined equations (2.2) and (2.5) has first-order accuracy.

3. A Two-Dimensional Example

3.1. *Preliminary*

The two-dimensional version of Richtmyer's scheme for the hyperbolic system of partial differential equations $w_t + f_x + h_y = 0$, is

$$w_{j,k}^{l+1} = \tfrac{1}{4}(w_{j+1,k}^l + w_{j-1,k}^l + w_{j,k+1}^l + w_{j,k-1}^l) - \tfrac{1}{2}\frac{\Delta t}{\Delta x}(f_{j+1,k}^l - f_{j-1,k}^l) - \tfrac{1}{2}\frac{\Delta t}{\Delta y}(h_{j,k+1}^l - h_{j,k-1}^l)$$

$$w_{j,k}^{l+2} = w_{j,k}^l - \frac{\Delta t}{\Delta x}(f_{j+1,k}^{l+1} - f_{j-1,k}^{l+1}) - \frac{\Delta t}{\Delta y}(h_{j,k+1}^{l+1} - h_{j,k-1}^{l+1}) . \tag{3.1}$$

Here the notation $w_{j,k}^{l} = w(j\Delta x, k\Delta y, l\Delta t)$ has been used.

Suppose that the matrices A and B denote the Jacobians of $f(w)$ and $h(w)$ respectively.

Substitution of the Fourier component $\hat{w}_{(q,r)} e^{i(qx+ry)}$ into difference scheme (3.1) with constant matrices A and B yields

$$\hat{w}_{(q,r)}^{l+2} = G(\Delta t, q, r)\,\hat{w}_{(q,r)}^{l},$$

where

$$G(\Delta t, q, r) = I - i(\cos\alpha + \cos\beta)\frac{\Delta t}{\Delta x}(A\sin\alpha + B\sin\beta) - 2\left\{\frac{\Delta t}{\Delta x}(A\sin\alpha + B\sin\beta)\right\}^2;$$

$$\alpha = q\Delta x, \quad \beta = r\Delta y.$$

G is called the amplification matrix again.

We apply scheme (3.1) to the two-dimensional equations of a polytropic gas,

$$\frac{\partial\rho}{\partial t} + \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} = 0$$

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left\{\frac{3-\gamma}{2}\frac{m^2}{\rho} - (1-\gamma)\left(E - \tfrac{1}{2}\frac{n^2}{\rho}\right)\right\} + \frac{\partial}{\partial y}\left(\frac{mn}{\rho}\right) = 0$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}\left(\frac{mn}{\rho}\right) + \frac{\partial}{\partial y}\left\{\frac{3-\gamma}{2}\frac{n^2}{\rho} - (1-\gamma)\left(E - \tfrac{1}{2}\frac{m^2}{\rho}\right)\right\} = 0$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left\{\frac{1-\gamma}{2}\frac{m}{\rho^2}(m^2+n^2) + \gamma\frac{mE}{\rho}\right\} + \frac{\partial}{\partial y}\left\{\frac{1-\gamma}{2}\frac{n}{\rho^2}(m^2+n^2) + \gamma\frac{nE}{\rho}\right\} = 0. \tag{3.2}$$

ρ is the density, E is the energy, i.e., the sum of internal and kinetic energy, and γ is the ratio of specific heats. m and n denote ρu and ρv respectively, where u and v are the horizontal and vertical flow speed. The pressure p has been eliminated by means of the relation

$$p = (\gamma - 1)\{E - (m^2 + n^2)/(2\rho)\}.$$

The four equations (3.2) represent the conservation of mass, momentum (two components) and energy, in that order. By means of a similarity transformation Richtmeyer and Morton [8] derived in the case $\Delta x = \Delta y$ for the eigenvalues g_ν of matrix G

$$|g_\nu|^2 = (1 - 2\mu_\nu^2)^2 + \mu_\nu^2(\cos\alpha + \cos\beta)^2 \qquad (\nu = 1, 2, 3, 4)$$

where

$$\mu_\nu = (\Delta t/\Delta x)\sqrt{\sin^2\alpha + \sin^2\beta}\begin{Bmatrix} u' \\ u' \\ u'+c \\ u'-c \end{Bmatrix} \tag{3.3}$$

and $u' = (u\sin\alpha + v\sin\beta)/\sqrt{\sin^2\alpha + \sin^2\beta}$; c is the local sound speed. The condition $(\Delta t/\Delta x)\cdot(c + \sqrt{u^2 + v^2}) < 1/\sqrt{2}$ appears to be necessary for stability.

From relation (3.3) it is clear that the difference scheme (3.1) has no dissipation if at least one of the μ_ν equals zero and in certain cases the difference scheme may be unstable then. This will occur with the calculation of a detached shock before a rectangular body moving with constant supersonic speed along its axis of symmetry. This stationary problem will be solved by means of a converging process using the non-stationary equations (3.2).

For this purpose we introduce dimensionless variables through the free-stream values of the density and sound speed, denoted by ρ_∞ and c_∞ in the upstream region before the shock:

$$\bar{\rho} = \rho/\rho_\infty, \quad \bar{m} = m/(\rho_\infty c_\infty), \quad \bar{n} = n/(\rho_\infty c_\infty), \quad \bar{E} = E/(\rho_\infty c_\infty^2).$$

Substitution of these new variables together with $\bar{x} = x/L$, $\bar{y} = y/L$ and $\bar{t} = tc_\infty/L$, where L

denotes some representative length, into the equations (3.2) does not change these equations. Below we shall omit the bars.

Because of the symmetry of the problem only the upper half of the body with its surrounding flow is taken into account.

We shall use constant values of Δx, Δy and Δt, whilst $\Delta x = \Delta y$.

3.2. *Initial Conditions*

Figure 3 shows a discontinuity that was initially prescribed to facilitate the process at the early stages. At the downstream side of the discontinuity the values of the variables ρ, m, n and E are calculated by using the Rankine–Hugoniot conditions. For the lower part, where the discontinuity is perpendicular to the axis of symmetry, these conditions are given by $[\boldsymbol{f}] = \boldsymbol{0}$, where [ ] denotes the jump across the discontinuity.

For the upper part the conditions are given by $[\boldsymbol{f}] - [\boldsymbol{h}] = \boldsymbol{0}$. The dimensionless free-stream values are

$$\rho_\infty = 1, \quad m_\infty = M_\infty, \qquad n_\infty = 0, \quad E_\infty = \tfrac{1}{2}M_\infty^2 + \frac{1}{\gamma(\gamma-1)}\,;$$

M is the local Mach number.

Therefore, together with the geometry of the body, only the values of M_∞ and γ are to be specified to determine the problem uniquely.

3.3. *Boundary Conditions*

Except for the left boundary, the boundary conditions are imposed by means of rows of virtual points (see Fig. 4).

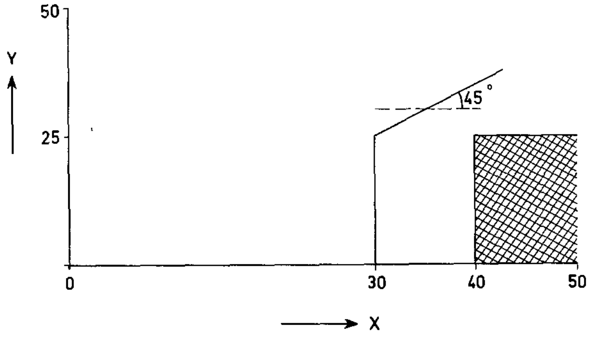

Figure 3. Initial discontinuity before the body.

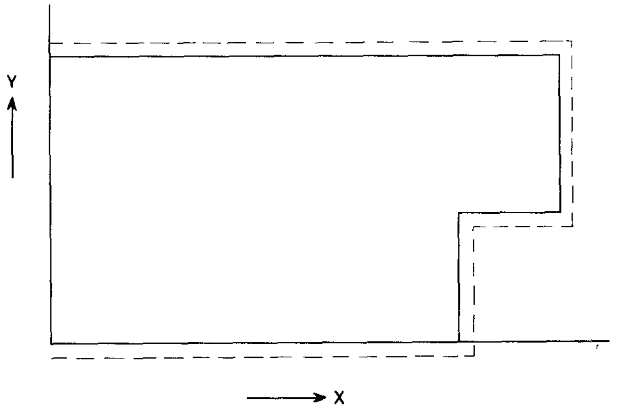

Figure 4. Boundaries with lines of virtual points.

At the axis of symmetry ($y=0$) we take as conditions at every time $t=(l+1)\Delta t$:

$$\rho_{j,-1}=\rho_{j,1}\,,\quad m_{j,-1}=m_{j,1}\,,\qquad n_{j,-1}=-n_{j,1}\,,\quad E_{j,-1}=E_{j,1}\,.$$

Similar conditions are prescribed on the upper boundary of the body, and, interchanging the conditions for m and n, on the front side of the body.

On the upstream boundary of the flow region ($x=0$) the variables will have the free-stream values.

On the downstream boundary and the upper boundary of the flow region we impose the conditions, representing the continuity of the first derivatives, in directions which are situated between the directions of the characteristics in the steady case (see Fig. 6). In this way, the influence of the condition which has been specified in some point of the boundary is directed into the domain of dependence of this point. On the downstream boundary ($x=J\Delta x$) we thus have at every time $t=(l+1)\Delta t$:

$$\rho_{J+1,k}=2\rho_{J,k}-\rho_{J-1,k}\,,$$

and the same condition for m, n and E.

On the upper boundary ($y=K\Delta y$) we have

$$\rho_{j+1,K+1}=2\rho_{j,K}-\rho_{j-1,K-1}\,,$$

and the same condition for m, n and E.

The upper boundary will be taken so far from the body that the lower characteristics proceeding from this boundary behind the shock do not intersect or touch the sonic line (see Fig. 6). In the stationary case, which is approximated here, this region is subsonic and a disturbance in any of its points would influence the whole region.

3.4. *Numerical Results*

The computation performed by applying scheme (3.1) to the system (3.2), together with the initial and boundary conditions described above, yields unstable results at the early stages. We observed an unlimited growth of short-wave components near the axis of symmetry: within the shock and at the body, and also at the top of the body near the corner.

Around the stagnation point we have $u\approx 0$ and $v\approx 0$, and thus $u'\approx 0$, whilst near the top of the body, i.e., at the lower part of the sonic line, $u\approx c$, and thus $u'\approx c$.

It follows that in both cases relation (3.3) yields $|g_v|\approx 1$ and no damping is present locally. This is especially true for the wave components for which is valid $\cos\alpha+\cos\beta=0$.

The following finite-difference smoothing operator appears to stabilize the solution

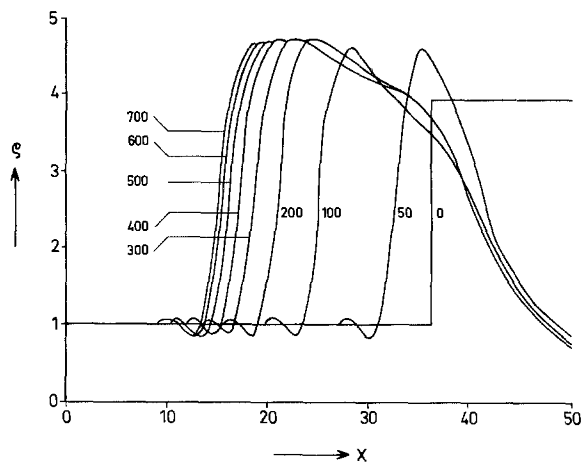

Figure 5. Densities along the coordinate $y=30$ at the times $t=0{,}50\,\Delta t$, $100\,\Delta t$, ..., $700\,\Delta t$. $M_\infty=4{,}3$, $\gamma=1{,}4$; $\Delta x=\Delta y=1$, $\Delta t=0{,}2$.

$$w'^{l+2}_{j,k} = \tfrac{1}{4}\left(w^{l+2}_{j+1,k} + w^{l+2}_{j-1,k} + w^{l+2}_{j,k+1} + w^{l+2}_{j,k-1}\right). \tag{3.4}$$

Substitution of the Fourier component $\hat{w}_{(q,r)}\, e^{i(qx+ry)}$ into (3.4) yields

$$\hat{w}'^{l+2}_{(q,r)} = \tfrac{1}{2}(\cos\alpha + \cos\beta)\,\hat{w}^{l+2}_{(q,r)},$$

with, as before, $\alpha = q\Delta x$ and $\beta = r\Delta y$. Again the multiplicative factor is real and damping occurs for the components with $\cos\alpha + \cos\beta = 0$.

Taylor expansions of equation (3.4) about the point $(j\Delta x, k\Delta y)$ yields

$$w' = w + \tfrac{1}{4}(\Delta x)^2 w_{xx} + \tfrac{1}{4}(\Delta y)^2 w_{yy} + 0\left[(\Delta x)^4, (\Delta y)^4\right],$$

and thus dissipation is introduced again.

Scheme (3.4) is applied after the two steps of equation (3.1) at every time $t = (l+2)\Delta t$ and a stable and convergent process is obtained (see Fig. 5).

The positions of shock and sonic line, together with the characteristics in two points of the upper and right boundaries are shown in Figure 6.

Densities along some lines y = constant have been plotted in Figure 7. Because of the smoothing element, the accuracy of our approximation is of first order, i.e. we have to deal with the

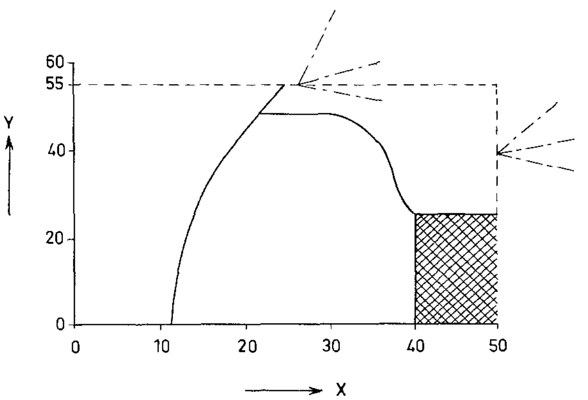

Figure 6. Shock, sonic line and characteristics in two boundary points ($t = 900\ \Delta t$).

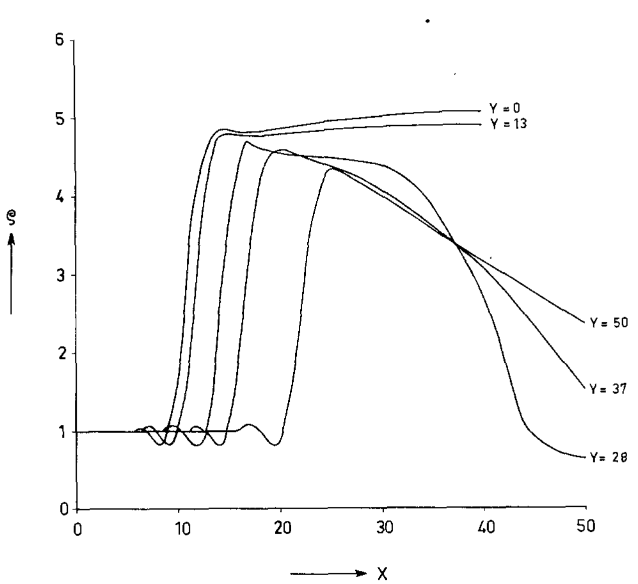

Figure 7. Density profiles along lines parallel to the axis of symmetry ($t = 900\ \Delta t$).

accuracy of Lax's scheme. This scheme consists of the first step of scheme (3.1), and applying Lax's scheme to this problem, which needs approximately the same computation time as (3.1) and (3.4) together, yields a shock which is spread out over about ten meshes, as against three or four meshes in the case of our approximation.

The smoothing described by Lapidus [4] does not completely stabilize our calculations.

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